Finite Mathematics

Sixth Edition

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McGraw Hill Education

Finite **MATHEMATICS**

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Contents

Preface

Probability and linear mathematics, the core of the traditional course in finite mathematics, provide some of the most basic and widely used mathematical tools in business, and the social and life sciences. These core topics and their applications are presented in Parts I and II of this text. Throughout the book there is an emphasis on ideas and techniques useful in constructing models and solving problems.

You learn to solve problems by working problems. Therefore, we include many exercises for you to use in developing and testing your problem-solving ability. Some are very easy, most are similar in difficulty to the discussions and the examples explained in the text, and a few are fairly tough. To solve problems, you must know where to start and how to proceed. We use discussions and examples to introduce and illustrate ideas and techniques to aid you in acquiring these skills. Some of the examples are straightforward computation, others show you how to solve problems by combining several ideas and techniques, and yet others illustrate the important method of breaking a problem into simpler problems, solving them one at a time, and then putting the results together to solve the original problem. Since we cannot provide examples of every type of problem and every setting you may encounter, we identify fundamental principles that should be helpful in unfamiliar situations.

Chapters 1 through 4 present basic concepts in probability, and it is common for this material to constitute about one-half of a one-semester finite mathematics course. The other half of such a course is usually devoted to linear equations, matrices, and linear programming, and these topics are covered in

PART

1

Probability Models

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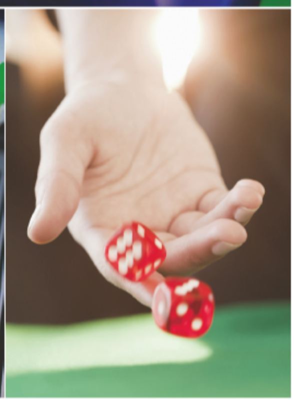

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CHAPTER 1

Sets, Partitions, and Tree Diagrams

1.0 THE SETTING AND OVERVIEW

We develop a common notation and terminology for sets and set operations which will be helpful in explaining and understanding probability. The discussion begins with sets, ways of combining sets, and a connection between set operations and certain logical operations. A useful technique of representing sets with diagrams is developed. We introduce a special type of set needed in our work on probability, the set of outcomes of an experiment. We develop three key methods for counting the elements in particular kinds of sets. These methods are partitions, tree diagrams, and the multiplication principle, concepts that will be applied and extended in Chapters 2, 3, and 4.

1.1 REVIEW OF SETS AND SET OPERATIONS

The students in a finite mathematics class form a set. So do the workers in an office, the books on a shelf, and the courses taken by a student this semester. A *set* is a collection whose members are specified by a list or a rule. The items in the collection are the *elements* of the set. To use a rule to specify a set, the rule must make it possible to determine precisely which things are in the set and which are not. When a set is specified by a list, the usual practice is to list the elements, each one exactly once, between a pair of braces, thus, the set S of names of states beginning with the letter A may be denoted by

$$S = \{\text{Alabama, Alaska, Arizona, Arkansas}\}$$

When a rule is used to specify a set, the usual practice is to write the rule after a symbol denoting a general element of the set followed by a colon and to include all the information in braces. For instance, to specify the set S with a rule, we can write

$$S = \{x\colon\ x \text{ is the name of a state beginning with the letter A}\}$$

In this expression the symbols ''$S = \{x\colon \ldots\}$'' are read ''S is the set of all x such that. . . .''

Whether a list or a rule is used to specify a set, it is important to remember that either something belongs to a set or it does not. It cannot partly belong to a set, and it cannot belong to a set several times.

Example 1.1 Consider the set $I = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. Here we have specified the set by listing all elements in the set. This can also be described in more than one way by using a rule. Two such ways are the following:

$$I = \{n : n \text{ is a positive integer that is not less than 2 or greater than 12}\}$$

$$I = \{s : s \text{ is the sum of the two numbers on top when two standard six-sided dice are rolled}\}$$
■

In some cases a list of the elements provides the simplest and most useful representation of a set. In other cases a rule is preferable. For sets with large numbers of elements, the use of a rule is often the only practical way to define the set.

To indicate that x is an element of a set X, we write $x \in X$. To indicate that x is *not* an element of X, we write $x \notin X$. Thus in Example 1.1 we have $8 \in I$, but $15 \notin I$.

A set A is a *subset* of a set B, written $A \subset B$, if every element in A is also in B. If $A \subset B$ and $B \subset A$, then A and B have exactly the same elements, and we say that A and B are *equal*. We write $A = B$.

To illustrate the notation, let $F = \{4, 8, 12\}$ and $G = \{1, 4, 6\}$, and let I be the set of Example 1.1. Then $F \subset I$ since every element of F is also an element of I. The set G is not a subset of I since not all elements of G are in I. Indeed, $4 \in I, 6 \in I$, but $1 \notin I$. In such a case it is sometimes convenient to write $G \not\subset I$.

It is often helpful to describe sets in terms of other sets. For instance, if

$$A = \{2, 4, 8, 12\}$$
$$B = \{2, 6, 10, 12, 14\}$$
and
$$C = \{2, 4, 6, 8, 10, 12, 14\}$$

then C is the set of all elements which are in A or in B or in both. Thus, we can view C as the set which results from combining A and B in a specific way. The operation which combines sets in this way is known as the *union*.

Let A and B be sets. The set $A \cup B$, called the *union* of A and B, consists of all elements which are in A or in B or in both.

$$A \cup B = \{x: x \in A \text{ or } x \in B\}$$

Note that in expressions such as ‘‘$x \in A$ or $x \in B$’’ the ‘‘or’’ is inclusive; that is, the condition is fulfilled if at least one of $x \in A$ or $x \in B$ holds. This includes the possibility that both hold.

To continue, let

$$A = \{2, 4, 8, 12\} \qquad B = \{2, 6, 10, 12, 19\} \qquad J = \{2, 12\}$$

Then J is the set of elements which are in both A and B. The operation which combines sets in this way is known as the *intersection*.

Let A and B be sets. The set $A \cap B$, called the *intersection* of A and B, consists of all elements which are in both A and B.

$$A \cap B = \{x: x \in A \text{ and } x \in B\}$$

Example 1.2 Let sets S, E, C, and M be defined as follows:

$$\begin{aligned} S &= \{\text{CT, MA, MD, CA, CO, MI, MN}\} \\ E &= \{\text{CT, MA, MD}\} \\ C &= \{\text{CA, CO, CT}\} \\ M &= \{\text{MA, MD, MI, MN}\} \end{aligned}$$

The elements of these sets are actually the standard abbreviations for names of states. The meaning is unimportant, however, and the elements can be viewed simply as symbols. Since each element of E is also an element of S, set E is a subset of S and we write $E \subset S$. Likewise, $C \subset S$ and $M \subset S$. However, since CT $\in E$ but CT $\notin M$, set E is not a subset of M, written $E \not\subset M$. Sets E and C have only the element CT in common, but sets E and M share the elements MA and MD. Therefore,

$$E \cap C = \{\text{CT}\} \qquad \text{and} \qquad E \cap M = \{\text{MA, MD}\}$$

Combining pairs of sets with the union operation, we have

$$\begin{aligned} E \cup C &= \{\text{CA, CO, CT, MA, MD}\} \\ E \cup M &= \{\text{CT, MA, MD, MI, MN}\} \\ \text{and} \qquad C \cup M &= \{\text{CT, MA, MD, CA, CO, MI, MN}\} = S \end{aligned}$$

■

Note that the element CT which appears in both E and C appears only once in $E \cup C$. As we noted earlier, when a set is specified by a list, each element in the set must appear in the list exactly once. Also, if a set is specified by a list, then the order in which the elements appear in the list does not matter. Thus {CT, MA, MD} is the same as {MA, MD, CT} or {MD, CT, MA}.

In Example 1.2 the sets $C = \{\text{CA, CO, CT}\}$ and $M = \{\text{MA, MD, MI, MN}\}$ have no elements in common. That is, the set $C \cap M$ has no elements. Likewise, the set

$$S = \{x\colon x \text{ is the name of a state beginning with the letter B}\}$$

has no elements.

The set which contains no elements is known as the *empty set*, and it is denoted by $\emptyset$. By convention the empty set is considered to be a subset of every set.

Since the empty set has no elements, we see that for every set A,

$$A \cap \emptyset = \emptyset \qquad \text{and} \qquad A \cup \emptyset = A$$

Two sets A and B are *disjoint* if $A \cap B = \emptyset$.

The sets $C = \{\text{CA, CO, CT}\}$ and $M = \{\text{MA, MD, MI, MN}\}$ are disjoint since $C \cap M = \emptyset$.

The definitions of union and intersection were formulated for two sets. With the use of parentheses they can be used in expressions which involve more than two sets. For instance, if we have three sets A, B, and C, then the set of all elements which are in A and in B and in C can be written as either $A \cap (B \cap C)$ or $(A \cap B) \cap C$. In this case the parentheses do not matter, and thus we can write $A \cap B \cap C$ for short. Likewise the set of all elements in A or in B or in C (or in more than one of these sets) can be unambiguously denoted by $A \cup B \cup C$. However, when an expression involves both union and intersection operations, it is generally necessary to use parentheses in writing the expression. The operations within the parentheses are to be carried out first.

Example 1.3 Let $A = \{a, b, c\}$, $B = \{a, c, e\}$, and $C = \{a, d\}$. Then

$$\begin{aligned} A \cap B \cap C &= \{a\} \\ A \cup B \cup C &= \{a, b, c, d, e\} \\ (A \cap B) \cup C &= \{a, c\} \cup \{a, d\} = \{a, c, d\} \\ A \cap (B \cup C) &= \{a, b, c\} \cap \{a, c, d, e\} = \{a, c\} \end{aligned}$$

Here we have $(A \cap B) \cup C \neq A \cap (B \cup C)$. The parentheses are clearly crucial to the meaning of the expressions. ■

We have defined the intersection of two sets A and B (the set of elements which are in A *and* B) and the union of A and B (the set of elements which are in A *or* B or both). Thus we have operations on sets which associate naturally with "and" and "or." We turn next to an operation on sets which is the natural associate of the word "not."

A set U is said to be a *universal set* for a problem if all sets being considered in the problem are subsets of U. Given a universal set U, the *complement* of a subset A of U is the set of all elements in U which are not in A. The complement of A is written A'.

$$A' = \{x, x \in U \text{ and } x \notin A\}$$

Notice that if A and B are subsets of U, then the set of elements in A which are not in B can be written as $A \cap B'$. Such sets arise frequently in applications.

Example 1.4 Let $U = \{\text{CA, CO, CT, IL, IN}\}$, $X = \{\text{CA, CT, IL}\}$, $Y = \{\text{CO, CT, IN}\}$, and $Z = \{\text{CO, IN}\}$. Then

$$X' = \{\text{CO, IN}\} = Z, \quad Y' = \{\text{CA, IL}\} \quad Z' = \{\text{CA, CT, IL}\} = X$$
$$Y \cap Z' = \{\text{CT}\} \quad X \cap Z' = \{\text{CA, CT, IL}\} = X \quad Z \cap Y' = \emptyset$$ ■

In addition to taking unions, intersections, and complements, there are other useful ways of building new sets from given ones. For example, suppose that a sociologist has enough money to conduct one survey. The survey can be conducted either by mail (M) or by phone (P) in one of three cities: Atlanta (A), Boston (B), or Cincinnati (C). Thus the choice for the sociologist can be viewed as selecting a method (M or P) and a city (A, B, or C). Each possible survey can be denoted by an *ordered pair* of elements, one from the set $\{M, P\}$ and one from the set $\{A, B, C\}$. Thus selecting a survey is clearly the same as selecting an element from the set

$$\{(M, A), (M, B), (M, C), (P, A), (P, B), (P, C)\}$$

The *cartesian product* of sets A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b): a \in A, b \in B\}$$

Example 1.5 Let

$$A = \{a, c, e\} \quad B = \{b, d, e\} \quad C = \{b, d\}$$

Then

$$\begin{aligned} A \times C &= \{(a, b), (a, d), (c, b), (c, d), (e, b), (e, d)\} \\ B \times C &= \{(b, b), (b, d), (d, b), (d, d), (e, b), (e, d)\} \\ C \times C &= \{(b, b), (b, d), (d, b), (d, d)\} \end{aligned}$$

■

Since $A \times C$ in Example 1.5 is a set, the order of the elements within the braces is *not* important. In particular, we could also write

$$A \times C = \{(a, b), (c, b), (e, b), (a, d), (c, d), (e, d)\}$$

The order of the symbols within the parentheses *is* important: The element (x, y) is different from (y, x) for x different from y. Our next example illustrates this.

Example 1.6 A division of a soccer league consists of four teams: the Argots (A), Bots (B), Cams (C), and Drams (D). Each game can be represented as an ordered pair of teams in which the first entry denotes the home team. With this notation the set of all possible games is a subset of the cartesian product of the set $L = \{A, B, C, D\}$ with itself, $L \times L$. Note that the cartesian product $L \times L$ contains elements such as (A, A), which are not legitimate games. In fact, the set of all possible games is

$$G = \{(A, B), (A, C), (A, D), (B, A), (B, C), (B, D), (C, A), (C, B), (C, D), (D, A), (D, B), (D, C)\}$$

and the set of all games involving the Argots, Bots, and Cams is

$$H = \{(A, B), (A, C), (B, A), (B, C), (C, A), (C, B)\}$$

■

Example 1.6 illustrates the importance of order in the construction of an ordered pair. For instance, the game between A and B with A as the home team is denoted by (A, B), while (B, A) denotes the game between the same teams with B as the home team. Thus, the ordered pair (A, B) is not the same as the ordered pair (B, A). However, as we noted earlier, the order of elements in the list specifying the set is unimportant, and set H can also be represented, e.g., as

$$H = \{(A, B), (B, A), (A, C), (C, A), (B, C), (C, B)\}$$

Exercises for Section 1.1

1. Let $R = \{a, b\}$, $S = \{a, c, f\}$, and $T = \{a, b, c, d, e\}$. Decide whether each of the following assertions is correct.
(a) $S \subset T$ (b) $R \subset S$ (c) $b \in R \cap T$
2. With R, S, and T defined as in Exercise 1, decide whether each of the following assertions is correct.
(a) $R \subset T$ (b) $(R \cup S) \subset T$ (c) $c \in S \cap T$
3. With R, S, and T defined as in Exercise 1, find $(R \cup S) \cap T$.
4. Let $U = \{1, 2, t, u, v, x, y, z\}$, $E = \{2, t, y\}$, $F = \{1, 2, u, y, z\}$, and $G = \{1, 2, u, y\}$. Find $(E \cup F) \cap G'$.
5. Let $A = \{p, q, r\}$. Find all nonempty subsets B and C of A such that $B \cap C = \emptyset$ and $B \cup C = A$.
6. Let $U = \{1, 2, 3, 4, 5, 6, 7\}$ be a universal set with subsets $A = \{1, 3, 5\}$ and $B = \{1, 5\}$. List the elements in each of the following sets.
(a) A' (b) $A' \cap B'$ (c) $A \cup B$ (d) $(A \cup B)'$
7. The sets M, A, B, and C are defined as follows:

$$M = \{\text{Minnesota, Michigan, Montana, Massachusetts}\}$$
$$A = \{\text{Alabama, Arkansas, Michigan}\}$$
$$B = \{\text{Montana, Michigan}\}$$
$$C = \{\text{Alabama, Arkansas}\}$$

Decide which of the following subset relationships are correct.
(a) $B \subset M$ (b) $B \subset C$ (c) $C \subset A$
(d) $C \subset B$ (e) $C \subset M$ (f) $A \subset (B \cup C)$
8. The sets R, S, and T are subsets of a universal set U. Which of the following always holds?
(a) $R \cap S \subset R$ (b) $T \subset T \cap \emptyset$
(c) $R \cup (S \cap T) \subset R \cap (S \cup T)$ (d) $R' \cup S' = (R \cup S)'$
9. Let $U = \{u, v, w, x, y, z, 1, 2, 3\}$, $E = \{2, y, w, z\}$, $F = \{2, 3, u, y, z\}$, and $G = \{1, 2, 3, w, y\}$. List the elements in each of the following sets.
(a) E' (b) $F \cup G'$ (c) $(E \cup F) \cap G'$
10. Let $U = \{x, y, z, 1, 2, 3\}$, $A = \{y, z, 2\}$, $B = \{y, 1, 2\}$, and $C = \{x, 3\}$. List the elements in each of the following sets.
(a) $A \cup B$ (b) $B \cap C$ (c) A'
(d) $(A \cup B) \cap (B \cup C)$ (e) $(B \cap A') \cap C'$
11. Let U, A, B, and C be defined by

$$U = \{a, b, c, 1, 2, 3\}$$
$$A = \{a, b, c\} \qquad B = \{a, 2, 3\} \qquad C = \{1, 2, 3\}$$

List the elements in each of the following sets.
(a) $A \cup B$ (b) $B \cap C$ (c) $(A \cup B) \cap (B \cup C)$
(d) A' (e) $A \cap B'$ (f) $A \cup C'$
12. Let $X = \{b, p, 4, 7\}$ and $Y = \{a, p, 4\}$ be subsets of a universal set $U = \{a, b, p, 1, 4, 7\}$. Which of the following are not true statements?
(a) $b \in X \cup Y$ (b) $\{p, 4\} = X \cap Y$ (c) $7 \in X \cap Y'$
(d) $1 \in X' \cap Y$ (e) $1 \in X' \cup Y$

13. Let sets A, B, C, and D be defined by

$$\begin{aligned} A &= \{x : x \text{ owns a GM car}\} \\ B &= \{x : x \text{ works for GM}\} \\ C &= \{x : x \text{ is the president of GM}\} \\ D &= \{x : x \text{ owns stock in GM}\} \end{aligned}$$

Describe in words each of the following sets.
(*a*) $A \cap B$ (*b*) $B \cap A'$ (*c*) $(A \cup B) \cap D$ (*d*) $C \cap A$

14. Let X and Y be sets with $a \in X$ and $b \in Y$. Is it *always* true (yes or no) that $\{a, b\} \subset X \cup Y$? That $\{a, b\} \subset X \cap Y$?

15. Let A and B be subsets of a universal set U. It is *always* true that
(*a*) $B \cap A' \subset A$ (*b*) $A \cap B \subset A \cup B$
(*c*) $A' \cap B' \subset (A \cap B)'$ (*d*) $A' \cup B' \subset (A \cup B)'$

16. Let A, B, C, and D be subsets of U with $A \subset B$ and $C \subset D$. Is it *always* true that
(*a*) $A \cap C \subset B \cap D$ (*b*) $A' \cap C' \subset B' \cap C'$

17. Let $U = \{2, 4, 8, 16, 32, 64\}$. Which of the following pairs of subsets A, B, of the universal set U, satisfy the condition: $A \cap B' = \{2, 16\}$
(*a*) $A = \{2, 8, 16\}$, $B = \{4, 8, 64\}$
(*b*) $A = \{2, 16, 32\}$, $B = \{4, 8, 64\}$
(*c*) $A = \{2, 16\}$, $B = \{4, 64\}$

18. Let $U = \{w, x, y, z\}$. Find examples of subsets A and B of U which satisfy the stated condition.
(*a*) $A \cup B = A$ (*b*) $A \cap B = A$
(*c*) $A \cap B' = A$ (*d*) $A \cap B' = B \cap A'$

19. Let $U = \{1, 2, 3, 4, x, y\}$ be a universal set with subsets $X = \{1, 2, 3, x, y\}$, $Y = \{2, 4, y\}$, and $Z = \{2, x\}$. Use intersections, unions, and complements to express each of the following sets in terms of X, Y, and Z.

$$A = \{2, y\} \qquad B = \{1, 3, y\} \qquad C = \{2, 4, x, y\}$$

20. With X, Y, Z, and U as in Exercise 19, use intersections, unions, and complements to express the set $\{2, x, y\}$ in terms of X, Y, and Z.

21. List all subsets of the following sets.
(*a*) $\{x\}$ (*b*) $\{x, y\}$ (*c*) $\{x, y, z\}$

22. Counting the empty set and the set itself, how many subsets does each of the following sets contain?
(*a*) $\{x\}$ (*b*) $\{x, y\}$ (*c*) $\{x, y, z\}$ (*d*) $\{w, x, y, z\}$
Is there a pattern? If so, what is the pattern? How many subsets does a set with seven elements contain?

23. Let $U = \{a, b, c, 2, 4, 6\}$ be a universal set with subsets X, Y, and Z. Suppose that $X \cup Y = \{b, c, 2, 4, 6\}$, $X \cap Y = \{b, 2, 4\}$, $Y' \cap Z' = \{a, c\}$, and $Z' = \{a, c, 2\}$. Find sets X, Y, and Z which satisfy these conditions.

24. If $A = \{r, s, t\}$ and $B = \{s, t, u\}$: list the elements in $A \times A$, $A \times B$, $B \times A$, and $B \times B$.

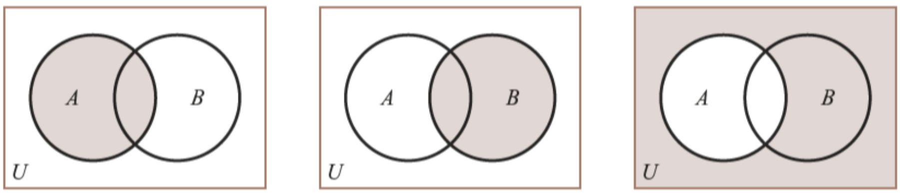
25. Let $A = \{a, b, c\}$ and $B = \{a, b, d\}$.
(*a*) List the elements in $A \times B$.
(*b*) List the elements in $(A \times B) \cap (B \times A)$.

26. Let $U = \{1, 2, 3, 4, 5, 6\}$ be a universal set with subsets X, Y, and Z. Suppose that $X \cup Y = \{1, 2, 4\}$, $Y \cap Z = \{4\}$, $(Y \cup Z)' = \{1, 3, 5\}$, $X \cap Y = \{4\}$, and $Z' = \{1, 2, 3, 5\}$. Find subsets X, Y, and Z.
27. Suppose $A \times B = \{(a, 1), (b, 1), (a, 2), (b, 2), (a, 3), (b, 3)\}$. Find A and B.
28. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 4\}$. Decide which of the following are correct.
    (*a*) $(A \times B) \cap (B \times A) = (A \times A) \cap (B \times B)$
    (*b*) $(A \times B) = \{(2,1), (1, 4), (1, 2), (3, 4), (3, 2), (1,1), (2, 2), (2, 4), (3, 1)\}$
    (*c*) $(A \times A) \subset ((A \times B) \cap (B \times A))$
    (*d*) $(A \times A) \subset ((A \times B) \cup (B \times A))$
29. Let $U = \{-2, -1, 0, 1, 2\}$ and $S = \{-1, 0, 1\}$. Also, let $A = \{(x, y) : x \in S, y = x^2\}$ and $B = \{(x, y) : x \in U, y = x^2\}$. Is it true that
    (*a*) $A \subset B$ (*b*) $A \subset S \times S$ (*c*) $B \subset S \times S$ (*d*) $B \subset U \times U$
30. Let A, B, S, and U be the sets given in Exercise 29. Find $A \times A$ and $B \times B$. Show that $A \times A \subset B \times B$. Is it always true that if $A \subset B$, then $A \times A \subset B \times B$? Why or why not?
31. Suppose $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$. If $C = \{d, e\}$, then $C \times A = ?$
32. Let $A = \{1, 2, 3, 4, 6, 8\}$ and $B = \{5, 6, 7, 8, 9\}$. A set W is defined to be the elements (pairs) of $B \times A$ for which at least one of the numbers is even. How many elements are there in W? How many elements are in $A \times W$?
33. Let $A = \{x, y, z, 1\}$ and $B = \{1, 2, 4\}$. If a set $U = (A \times B) \cup (B \times A)$, how many elements are there in U?
34. The set $F = \{1, 2, 3, 5, 8, 13, 21, 34\}$. Describe the set F by two different rules.
35. A set S has 6 elements, two of them being a and b. How many subsets of S do not include a or b? How many include a, but not b?

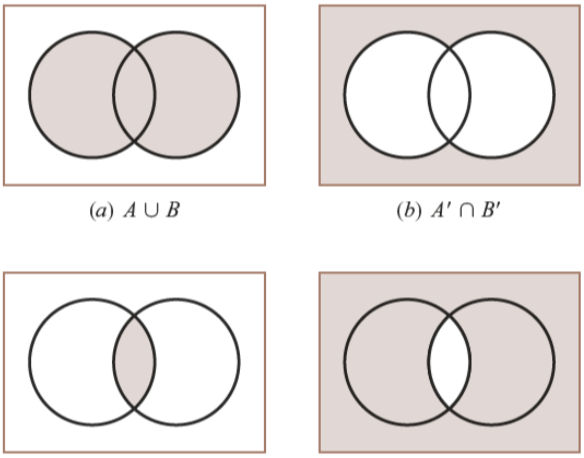
1.2 VENN DIAGRAMS AND PARTITIONS

In working with sets and the relations between sets, it is often helpful to represent them with diagrams or pictures. A *Venn diagram* serves this purpose. In a Venn diagram, a universal set U and its subsets are pictured by using geometric shapes. By convention the set U is usually represented by a rectangle, and the subsets of U are usually circles inside the rectangle. For example, subset A of U is shown as the shaded region in Figure 1.1*a*, subset B is shown in Figure 1.1*b*, and subset A' is shown in Figure 1.1*c*.

FIGURE 1.1



(*a*) A (*b*) B (*c*) A'



(*a*) $A \cup B$ (*b*) $A' \cap B'$ (*c*) $A \cap B$ (*d*) $A' \cup B'$

FIGURE 1.2

Subsets of U obtained by taking unions, intersections, and complements of A or B or both can also be represented by Venn diagrams. For instance, $A \cup B$ is illustrated in Figure 1.2*a*, and $A \cap B$ is illustrated in Figure 1.2*c*.

Two useful set equalities are known as *deMorgan's laws*.

For any subsets A and B of a universal set U

$$(A \cup B)' = A' \cap B'$$
$$(A \cap B)' = A' \cup B'$$

These equalities are illustrated in Figure 1.2. First, Figure 1.2*a* and *b* illustrate that $(A \cup B)' = A' \cap B'$. This relation can be read ''The complement of a union is the intersection of the complements,'' and it follows from the definitions of union, intersection, and complement. Likewise, Figure 1.2*c* and *d* illustrate that $(A \cap B)' = A' \cup B'$, which can be read ''The complement of an intersection is the union of the complements.''

Other useful relations can be illustrated with Venn diagrams and verified by using the definitions. Among these relations are the following distributive laws:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

These relations hold for any three sets A, B, and C. As we mentioned earlier, the parentheses are essential, and the expressions would be ambiguous without them.

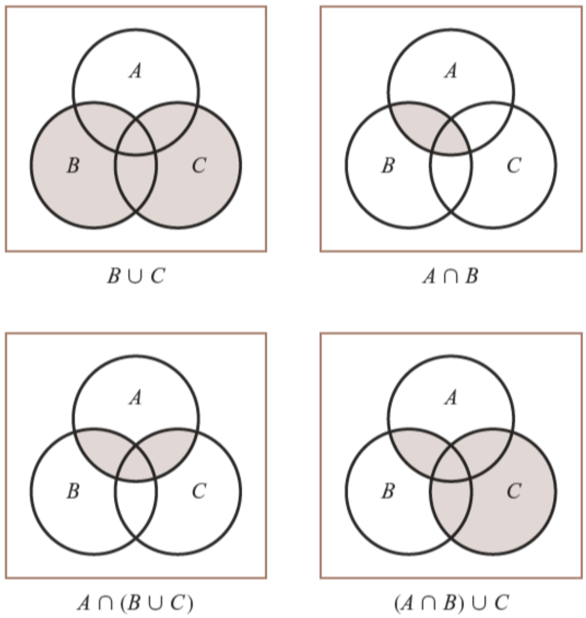
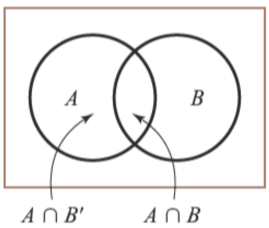


FIGURE 1.3

Example 1.7 Figure 1.3 illustrates that in general the sets $A \cap (B \cup C)$ and $(A \cap B) \cup C$ are different. ■

FIGURE 1.4



Venn diagrams provide us with a geometric way to represent the decomposition of a set into subsets. For example, in Figure 1.4 we illustrate how the set A can be decomposed into subsets $A \cap B$ (the football-shaped region) and $A \cap B'$ (the crescent-shaped region). The notion of decomposition of a set into subsets is extremely useful in the study of probability, and we consider the situation in greater detail.

Example 1.8 Let $X = \{2, 4, 6, 8, 10, 11, 12, 13, 14, 16, 18\}$.

Problem Find subsets X_1, X_2, and X_3 such that X_1 contains all even numbers in X less than 10, X_2 contains all odd numbers in X, and X_3 contains all even numbers in X at least as large as 10.

Solution

$$X_1 = \{2, 4, 6, 8\}, \qquad X_2 = \{11, 13\}, \qquad X_3 = \{10, 12, 14, 16, 18\}$$ ■

Note that $X = X_1 \cup X_2 \cup X_3$ and $X_1 \cap X_2 = \emptyset, X_1 \cap X_3 = \emptyset, X_2 \cap X_3 = \emptyset$.

Example 1.9 Let $U = \{\text{Cairo, Copenhagen, Lima, Paris, Rio de Janeiro, Vienna}\}$.

Problem Find subsets A, E, and S of U such that

$$\begin{aligned} A &= \{x : \ x \text{ is a city in Africa}\} \\ E &= \{x : \ x \text{ is a city in Europe}\} \\ S &= \{x : \ x \text{ is a city in South America}\} \end{aligned}$$

and find the union of these three sets and the intersection of each pair of them.

Solution The sets A, E, and S are the following: $A = \{\text{Cairo}\}$, $E = \{\text{Copenhagen, Paris, Vienna}\}$, and $S = \{\text{Lima, Rio de Janeiro}\}$. Then A, E, and S satisfy the following:

$$A \cup E \cup S = U \qquad A \cap E = \emptyset \qquad A \cap S = \emptyset \qquad E \cap S = \emptyset \tag{1.1}$$

■

In Example 1.9 sets A and E are disjoint. Likewise sets A and S are disjoint, and sets E and S are disjoint. Relationships like this occur frequently enough for us to use a special expression to describe them.

> The sets in a collection are said to be *pairwise disjoint* if every pair of sets in the collection is disjoint.

Also in Example 1.9, the result of each classification assigns a city to A, E, or S; the union $A \cup E \cup S$ is the set of all cities to be considered: $A \cup E \cup S = U$. Subsets A, E, and S of U which satisfy condition (1.1) form a *partition* of U. In general, a partition is the result of cutting up a set into subsets; each subset contains some elements of the set, and no two subsets can overlap. Formally,

> A *partition* of a set U is a collection of nonempty subsets of U which are pairwise disjoint and whose union is the entire set U.

In the use of sets in finite mathematics, one of our main concerns is to count the number of elements in certain sets. This is the primary topic of the next section, and it will recur frequently in our work on probability. Partitions are especially useful in helping us count the number of elements in a set. To show how, we need some notation.

Let A be a set with a finite number of elements. The number of elements in A is denoted by $n(A)$.

For instance, if X, A, E, and S are the sets of Example 1.9, then $n(X) = 6$, $n(A) = 1$, $n(E) = 3$, and $n(S) = 2$. In this case we have

$$6 = n(X) = n(A) + n(E) + n(S) = 1 + 3 + 2$$

In fact, the definitions of a partition and the number of elements in a set lead to the following useful principle:

Partition Principle

If a set X is partitioned into subsets $X_1, X_2, \ldots, X_k$, then

$$n(X) = n(X_1) + n(X_2) + \cdots + n(X_k) \tag{1.2}$$

If each of the subsets $X_1, X_2, \ldots, X_k$ has the same number of elements, then Equation (1.2) can be simplified to

$$n(X) = kn(X_1) \tag{1.3}$$

This version of (1.2) will be useful when X is a cartesian product.

Example 1.10 A student is to plan a schedule which consists of one science course and one humanities course. The student can choose a science course in astronomy (A), biology (B), or chemistry (C) and a humanities course in history (H), philosophy (P), religion (R), or theater (T).

Problem Determine the number of possible schedules.

Solution A schedule is a science course (A, B, or C) *and* a humanities course (H, P, R, or T). Thus, a class schedule can be represented as an ordered pair in which the first entry is a science course and the second entry is a humanities course. The set of class schedules S is a cartesian product of the set of science courses $X = \{A, B, C\}$ and the set of humanities courses $Y = \{H, P, R, T\}$. This cartesian product can be arranged in the array

$$\begin{array}{cccc} (A, H) & (A, P) & (A, R) & (A, T) \\ (B, H) & (B, P) & (B, R) & (B, T) \\ (C, H) & (C, P) & (C, R) & (C, T) \end{array}$$

There are three rows in the array, one corresponding to each element in the set $\{A, B, C\}$; and there are four columns in the array, one corresponding to each element in the set $\{H, P, R, T\}$. We can view the set S as partitioned into three subsets, one corresponding to each row of the array. Each row of the array contains four elements, and we conclude from (1.3) that

$$n(S) = n(\{A, B, C\}) \cdot n(\{H, P, R, T\}) = 3 \cdot 4 = 12$$ ■

The technique used in Example 1.10 for counting the number of elements in a set which can be represented as the cartesian product of two sets is perfectly general. We have the following rule.

> If A and B are sets, then
>
> $$n(A \times B) = n(A) \cdot n(B) \tag{1.4}$$

As we shall see in our next example, it is often necessary to consider cartesian products of more than two sets. The definition is similar to the definition of the cartesian product of two sets. For instance, the cartesian product $E \times F \times G$ of three sets is the set of all ordered triples (e, f, g), with $e \in E, f \in F$, and $g \in G$.

Example 1.11 Suppose that the student of Example 1.10 also plans to take one language course, either French (F) or German (G). Then the possible class schedules can be represented by ordered triples (x, y, z) where $x \in \{A, B, C\}$, $y \in \{H, P, R, T\}$, and $z \in \{F, G\}$. That is,

$$S = \{A, B, C\} \times \{H, P, R, T\} \times \{F, G\}$$

The number of class schedules available to the student is $n(S)$. Hence, as in Example 1.10, $n(S)$ can be obtained by taking the product of the numbers of elements in each of the sets in the cartesian product:

$$n(S) = n(\{A, B, C\}) \cdot n(\{H, P, R, T\}) \cdot n(\{F, G\}) = 3 \cdot 4 \cdot 2 = 24$$ ■

The result illustrated in Example 1.11 for three sets can be extended to an arbitrary number of sets. The corresponding general result is as follows:

> If $X_1, X_2, \ldots, X_k$ are sets, then
>
> $$n(X_1 \times X_2 \times \cdots \times X_k) = n(X_1) \cdot n(X_2) \cdots n(X_k)$$

Example 1.12 Suppose X_1 and X_2 form a partition of X, and Y_1, Y_2 form a partition of Y.

Problem If $n(X_1) = 4$, $n(X_2) = 5$, $n(Y_1) = 3$, and $n(Y_2) = 6$, find $n(X_1 \times X_2 \times Y)$.

Solution Since $n(Y) = n(Y_1 \cup Y_2) = 9$, we have $n(X_1 \times X_2 \times Y) = 4 \cdot 5 \cdot 9 = 180$. ■

Exercises for Section 1.2

1. Let A, B, and C be subsets of a set U. Draw a Venn diagram to illustrate each of the following sets. In each case, shade the area corresponding to the designated set.
   (a) $A' \cap B$ (b) $A \cup B'$
   (c) $A \cup B \cup C$ (d) $(A \cup B) \cap C'$
2. In each case determine which of points v, w, x, y, and z in Figure 1.5 are contained in the specified set.
   (a) $A \cap C'$ (b) $A \cup C'$
   (c) $A \cup (B \cap C)$ (d) $(B \cap C)'$
3. In each case determine which of points v, w, x, y, and z in Figure 1.5 are contained in the specified set.
   (a) $A \cup C$ (b) $A \cap B$
   (c) $A \cup B$ (d) $B \cap C$
4. Using Figure 1.5, decide which of the following statements are true and which are false.
   (a) $z \in A \cup C'$ (b) $y \in B \cup (A \cap C')$
   (c) $y \in (B \cup C) \cap A'$ (d) $v \in (B \cup C) \cap (A \cup C)$
5. Using Figure 1.5, decide which of the following statements are true and which are false.
   (a) $\{x, y\} \subset A \cap B \cap C$ (b) $\{v, y, z\} \subset (A \cap C) \cup B$
   (c) $\{w, x, y\} \subset (A \cup B) \cap C$ (d) $\{y, z, v\} \subset (A \cup B) \cap C$
6. Describe the shaded areas in each Venn diagram of Figure 1.6, by using the set operations of union, intersection, and complement and the sets A, B, and C.

FIGURE 1.5

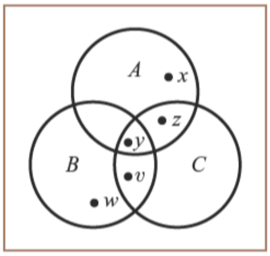
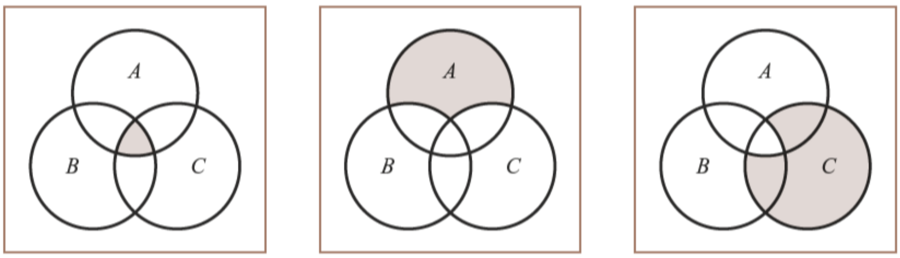
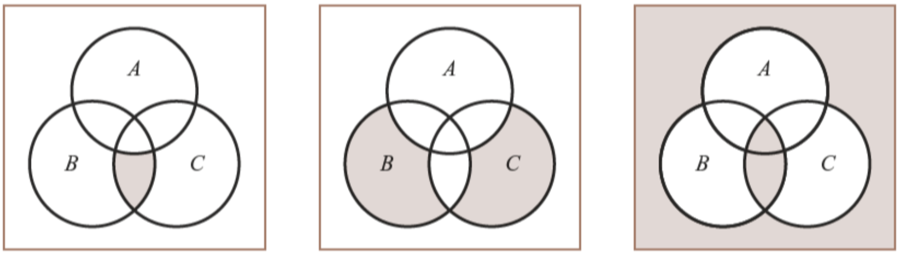


FIGURE 1.6



(a) (b) (c)

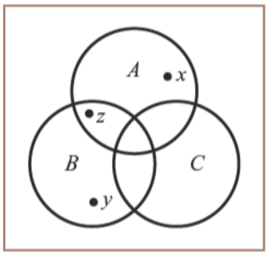


(a) (b) (c)

FIGURE 1.7

7. Repeat Exercise 6, using Figure 1.7.
   (a) $x \in A \cap C$ (b) $z \in (A \cap B)' \cup C$ (c) $y \in B'$
   (d) $x \in C'$ (e) $y \in A' \cup C$
8. Decide which of the following are ‘‘always true’’, ‘‘sometimes true’’, or ‘‘never true’’.
   (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   (b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (c) $(A \cup B') \cap C' = (A \cup C') \cap B'$
9. Determine which (if any) of the following set relations are true for *all* sets A, B, and C. (*Hint*: Use Venn diagrams.)
   (a) $A' \cap B' = (A \cap B)'$ (b) $(A \cap B') \subset A'$
   (c) $A' \cap B' \subset (A \cup B)'$ (d) $(A \cap B)' \subset A'$
   (e) $(A \cup B)' \cap C = (A' \cap B') \cup C$ (f) $(A \cap B)' \cup C' = (A' \cup B') \cup C'$

FIGURE 1.8

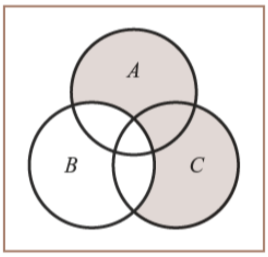


10. Which of the following is a true statement about the Venn diagram shown in Figure 1.8?
    (a) $x \in A \cap C$ (b) $z \in (A \cap B)' \cup C$ (c) $y \in B'$
    (d) $x \in C'$ (e) $y \in A' \cup C$
11. Let U be a universal set with disjoint subsets A and B; $n(U) = 60$, $n(A) = 25$, and $n(B) = 30$. Find $n((A \cup B)')$.
12. Let A and B be disjoint subsets in a Universal set U with $n(U) = 50$, $n(A \cup B) = 35$, and $n(B') = 25$. Find $n(A)$.
13. Let U be a universal set with disjoint subsets A and B; $n(A) = 25$, $n(A') = 40$, and $n(B') = 30$. Find $n(A \cup B)$.
14. Let $n(X \times Y) = 24$, $n(X \times Z) = 15$, and $n(Y \times Z) = 40$. Find $n(X \times Y \times Z)$.
15. Let X, A, B, and C be defined by

$$X = \{a, b, c, 1, 2, 3\}$$
$$A = \{a, b, c\} \qquad B = \{a, 2, 3\} \qquad C = \{1, 2, 3\}$$

    Which of the following pairs of subsets form a partition of X?
    (a) A and B (b) A and C
    (c) B and C (d) $(A \cup B)$ and $(C \cap B')$

FIGURE 1.9



16. The shaded region in Figure 1.9 is properly described by which two of the following:
    (a) $(A \cup C) \cap B'$ (b) $(A \cup C) \cap (B' \cup C)$
    (c) $(B' \cap C) \cup (B' \cap A)$ (d) $(A \cup B' \cup C)$

17. Let $A = \{1, 2, 3\}$ and $B = \{v, w\}$. By listing the elements in each of the three sets, show that $A \times \{v\}$ and $A \times \{w\}$ provide a partition of $A \times B$.
18. A set P is partitioned into subsets P_1, P_2, P_3. The number of elements in P_2 is 5 times the number in P_1 and the number of elements in P_3 is twice the number in P_1. If $n(P) = 40$, find $n(P_2)$.
19. Let $A = \{1, 7, 3, a, b\}$ and $B = \{3, c\}$. Determine
(*a*) $n(A \times B)$ (*b*) $n(B \times B \times B)$
20. A set U with $n(U) = 8$ is partitioned into 4 nonempty subsets A, B, C, and D. If all four sets are pairwise disjoint, then which of the following statements *must* be true?
(*a*) $n(A) + n(B) = n(C) + n(D)$ (*b*) $n(A) + n(B) \neq n(C) + n(D)$
(*c*) $n(A) + n(B) \geq 2$ (*d*) $n(A) + n(B) + n(C) + n(D) \neq 8$
21. A sociologist has a project which involves the collection of data. She is interested in data which can be obtained by mail, by phone, or in person in any of the cities of Atlanta, Boston, Chicago, Denver, or Elmira. She has funds for one project, that is, to collect data in one way from one city. Determine the number of possible ways to carry out the project.
22. Let A, B, and C be subsets of a universal set U with A and B disjoint, $n(U) = 110$, $n(A) = 35$, $n(B) = 44$, $n(A \cup B \cup C) = 96$, and $n((A \cup B) \cap C) = 28$. Find $n(C)$.
23. Let A, B, and C be distinct subsets of a universal set U with $A \subset B \subset C$. Also suppose $n(A) = 3$ and $n(C) = 7$. In how many different ways can you select B such that $n(B) = 4$? Repeat this exercise with $n(B) = 5$.
24. Suppose $n(A) = 5$, $n(B) = 10$, and $n(C) = 20$. Which of the following sets has more elements, $A \times B \times C$ or $B \times B \times B$?
25. A bag contains 6 green balls, 1 yellow ball, and 2 red balls. An experiment consists of selecting three balls, one after another without replacement, and noting the color of each ball selected. Suppose the set X of outcomes is partitioned into X_1, X_2, and X_3 where X_1 is the set of outcomes containing no red balls, X_2 is the set of outcomes containing 1 red ball, and X_3 is the set of outcomes containing 2 red balls. Find $n(X_1)$, $n(X_2)$, and $n(X_3)$.
26. Let A, B, and C be subsets of a universal set U with A and B disjoint, $n(U) = 120$, $n(A) = 35$, $n(B) = 44$, $n(A \cup B \cup C) = 100$, and $n((A \cup B) \cap C) = 28$.
(*a*) Find $n(C')$. (*b*) Find $n(C \cap A' \cap B')$.
27. A set X is partitioned into subsets X_1, X_2, and X_3. The number of elements in X_1 is twice the number in X_2, and the number in X_3 is 5 times the number in X_2. If $n(X) = 40$, find $n(X_1)$, $n(X_2)$, and $n(X_3)$.
28. A set X with $n(X) = 45$ is partitioned into three subsets X_1, X_2, and X_3. If $n(X_2) = 2n(X_1)$ and $n(X_3) = 3n(X_2)$, find the number of elements in subset X_1.
29. A set X with $n(X) = 60$ is partitioned into subsets $X_1, \ldots, X_6$. If $n(X_1) = n(X_2) = n(X_3)$, $n(X_4) = n(X_5) = n(X_6)$ and $n(X_1) = 4n(X_4)$, find $n(X_1)$.
30. Let A_1 and A_2 be a partition of A, and let B_1 and B_2 be a partition of B. Is it true that $A_1 \times B_1$, $A_1 \times B_2$, $A_2 \times B_1$, and $A_2 \times B_2$ form a partition of $A \times B$? Why or why not?

31. A set X with $n(X) = 100$ is partitioned into subsets A, B, C, D, and E. Suppose $n(B) = 3n(A)$, $n(C) = 4n(A)$, $n(E) = n(B) + n(C)$, and $n(D) = n(E) - 10$. Find the number of elements in subset D.
32. A set X with $n(X) = 120$ is partitioned into five subsets X_1, X_2, X_3, X_4, and X_5. If $n(X_5) = 5n(X_1)$, $n(X_4) = 2n(X_2)$, $n(X_2) = 2n(X_1)$, and $n(X_3) = n(X_2) + 8$, find $n(X_5)$.
33. A universal set U has subsets A, B, C, and D. It is known that $n(U) = 100$, $n(A) = n(B) = 30$, $n(C) = n(D) = 80$ and $n(A \cup B \cup C \cup D) = 80$. Find $n((A \cup B \cup C) \cap D)$.

1.3 SIZES OF SETS

We have seen that "the whole is equal to the sum of the parts" when we are dealing with subsets which form a partition of a set. This is summarized in the partition principle, formula (1.2). What if the sets of interest do not form a partition of another set? For instance, what if they are not disjoint? In such cases Venn diagrams and the partition principle are still useful when applied appropriately. We begin by analyzing a specific example in some detail. Our goal is both a technique and a very useful formula.

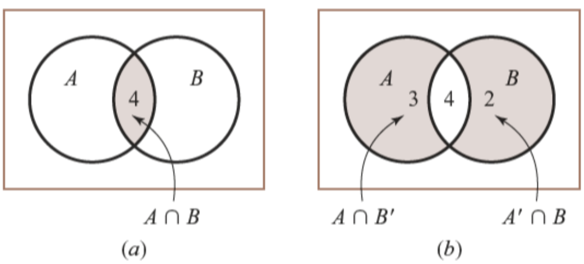
Example 1.13 A set U with nondisjoint subsets A and B has the following:

$$n(U) = 10 \qquad n(A) = 7 \qquad n(B) = 6 \qquad n(A \cap B) = 4$$

Problem Find $n(A \cup B)$.

Solution We use a Venn diagram with universal set U and subsets A and B. Inside subset $A \cap B$ we insert the number 4 to indicate that $n(A \cap B) = 4$. At this stage we have the diagram shown in Figure 1.10*a*. Next, since A has 7 elements and since 4 of them are in $A \cap B$, the portion of A not in $A \cap B$, that is, $A \cap B'$, must contain $7 - 4 = 3$ elements. We insert a 3 in the set $A \cap B'$ to indicate this. Likewise, since $n(B) = 6$, there must be 2 elements in $A' \cap B$. The information $n(A \cap B) = 4$, $n(A \cap B') = 3$, and $n(A' \cap B) = 2$ is shown

FIGURE 1.10



in Figure 1.10b. Using this information, we see from the partition principle [formula (1.2)] that $n(A \cup B) = 3 + 4 + 2 = 9$. ■

It is helpful to examine this example more closely. Since $n(A \cup B) = 9$ while $n(A) + n(B) = 6 + 7 = 13$, it is clear that in general we cannot obtain $n(A \cup B)$ simply by adding $n(A)$ and $n(B)$. In fact, by examining Figure 1.10 we see that adding $n(A)$ and $n(B)$ actually counts the elements in $A \cap B$ twice. It follows that to find $n(A \cup B)$, we must subtract $n(A \cap B)$ from $n(A) + n(B)$. In this way each element in $A \cup B$ will be counted exactly once. In Example 1.13 we have

$$n(A \cup B) = (3 + 4) + (4 + 2) - 4 = 9 = n(A) + n(B) - n(A \cap B)$$

Our argument holds for any two sets A and B. We have the useful formula

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) \tag{1.5}$$

Example 1.14 The webmaster for a new web site asked 100 recent visitors to the site about their impressions of the appearance and functionality of the home page for the site. In particular, she asked the following two questions:

(a) Do you prefer the simple appearance now used or would you prefer to have more information and options on the home page?

(b) Do you prefer a home page with no product ads or would you like to see ads for special sales?

The results of the survey are that 60 prefer the current simple home page, 45 prefer not to have ads for sales on the home page, and 25 prefer both a simple home page and not to have ads for sales.

Problem How many of the 100 responders to the survey prefer both a more informative home page and ads for sale items?

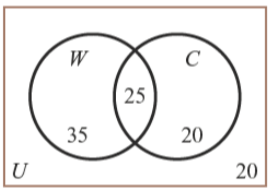


FIGURE 1.11

Solution We use a Venn diagram with a universal set U consisting of the 100 individuals who were surveyed by the webmaster. We also let W denote the subset of 60 people who prefer a simple home page, and we let C denote the subset of 45 people who prefer to not have sales ads on the home page. From the data of the problem we know that $n(W \cap C) = 25$. Since $n(W) = 60$, and $n(C) = 45$, there must be $60 - 25 = 35$ individuals in $W \cap C'$ and $45 - 25 = 20$ individuals in $W' \cap C$. Thus, we have the diagram and numbers shown in Figure 1.11.

From Figure 1.11 we see that $n(W \cup C) = 35 + 25 + 20 = 80$ and $n((W \cup C)') = 100 - 80 = 20$. We are interested in the individuals who are in W' (prefer more information on home page) and who are also in C'

(prefer sales ads on the home page), and therefore are in $W' \cap C'$. Since $W' \cap C' = (W \cup C)'$, the answer to the problem is

$$n(W' \cap C') = n((W \cup C)') = 20$$ ■

In Example 1.14 we used the fact that the number of elements in $(W \cup C)'$ is the difference between the number of elements in $W \cup C$ and the number of elements in the universal set U. This result holds for any subset A of U, and we have the formula

$$n(A') = n(U) - n(A) \tag{1.6}$$

The question raised in Example 1.14 can also be answered directly (without using a Venn diagram) by applying formulas (1.5) and (1.6). Proceeding in this way, we have

$$\begin{aligned} n(W \cup C) &= n(W) + n(C) - n(W \cap C) \\ &= 60 + 45 - 25 \\ &= 80 \end{aligned}$$

and therefore,

$$\begin{aligned} n[(W \cup C)'] &= n(U) - n(W \cup C) \\ &= 100 - 80 \\ &= 20 \end{aligned}$$

Nevertheless, *in general, it is best to draw the Venn diagram.* The diagram is a useful aid in organizing the information of the problem, and it helps us to spot mistakes which may result from using the formulas incorrectly. Also, the logic followed in using a Venn diagram is the same when there are three or more subsets of interest as when there are only two. This is illustrated in the following example.

Example 1.15 The GetFitFast Company requires each of its employees to pass a yearly physical examination. The results of the most recent examination of 50 employees were that 30 employees were overweight, 25 had high blood pressure, and 20 had a high cholesterol count. Moreover, 15 of the overweight employees also had high blood pressure, and 10 of those with a high cholesterol count were also overweight. Of the 25 with high blood pressure, there were 12 who also had a high cholesterol count. Finally, there were 5 employees who had all three of these undesirable conditions. When the reports reached the desk of the president, Jox Chinup, he asked, ''Don't we have any completely healthy employees around here?''